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Convergence Theorems for Integral Polynomial Approximations

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INTRODUCTION

Let $C(a, b)$ be the set of all real continuous functions on $[a, b]$ where $b - a < 4$ (see [6; 12, note 4.4]), let $f \in C(a, b)$ be arbitrary but fixed, and let $\| f - Q \|$ denote a measure of deviation or error in approximating f by $Q \in Q(Z)$, the set of all polynomials with integral coefficients.

DEFINITION 1. (a) f is approximable if and only if for each $\gamma > 0$ there exists $Q_\gamma \in Q(Z)$ such that $| f(x) - Q_\gamma(x)| < \gamma$ for all $x \in [a, b]$.

(b) f is matchable on S if and only if $S \subseteq [a, b]$, and there exists $Q \in Q(Z)$ such that $Q(x) = f(x)$ for all $x \in S$.

(c) Let $U(a, b) = \{Q \mid Q \in Q(Z), 0 \leqslant Q(x) < 1 \text{ for all } x \in [a, b], Q \not\equiv 0\}$. Then $U(a, b) \neq \varnothing$ (follows from [6, Theorem XIV]). Let $J(a, b) = \{x \mid x \in [a, b], Q(x) = 0 \text{ for all } Q \in U(a, b)\}$. The points of $J(a, b)$ are called the critical points of $[a, b]$.

The problem of the approximability of f has a fairly long history (see [3; 4; 6; 8; 12; 13; 15–17]); for numerous results closely related to this problem see [5; 7; 9; 10; 14]. More recently, the questions of existence and uniqueness of best approximations have been studied (see [1; 2; 11]). Extensive use of some of these results will be made in this paper.

Our main result (Theorem 1) states that if f is not approximable, it can be modified by changing its definition on the finite set $J(a, b)$, so as to be pointwise approximable by polynomials $Q \in Q(Z)$.

We prove also some results concerning approximation by polynomials of $Q(Z)$ in nonChebyshev norms.

CONVERGENCE THEOREMS AND BEST APPROXIMATIONS

The first norm to be considered is the maximum (Chebyshev) norm

$$\|f - Q\|_\infty = \max_{a \leqslant x \leqslant b} |f(x) - Q(x)|.$$

THEOREM 1. *Suppose f is not approximable, and set*

$$k_0 = \inf_{Q \in Q(Z)} \|f - Q\|;$$

then there exists a sequence $\{Q_l\}$ *in* $Q(Z)$ *such that* $Q_l \to F$ *in* $[a, b]$, *where*

$$F(x) = \begin{cases} f(x) & \text{for } x \in [a, b] - J(a, b), \\ T(x) & \text{for } x \in J(a, b) \text{ (which is } \neq \phi \text{ and finite [12]).} \end{cases}$$

Here $T \in Q(Z)$ *and is of degree* $< n$, *and we have* $\max_{J(a,b)} |f(x) - T(x)| = k_0$.

Proof. There exists $\gamma > 0$ such that for each $Q \in Q(Z)$, $|f(x) - Q(x)| > \gamma$ for some $x \in J(a, b) \neq \varnothing$ [2, Theorem 7]. From Theorem 5 of [2] it follows that the supremum of the set of these γ's is k_0.

Now let $\{\gamma_j\}$ be a real sequence such that $k_0 < \gamma_j \to k_0$. Construct corresponding sequences $\{Q_j\}$ and $\{x_j\}$ such that $x_j \in J(a, b)$, and $\gamma_j > |f(x_j) - Q_j(x_j)| \geqslant |f(x) - Q_j(x)|$ for all $x \in J(a, b)$. Since $J(a, b)$ is finite, there exists a subsequence $\{j'\}$ of $\{j\}$ such that $\{x_{j'}\}$ is a constant sequence (denote it by $\{x^*\}$). Then $\gamma_{j'} > |f(x_{j'}) - Q_{j'}(x_{j'})| \equiv |f(x^*) - Q_{j'}(x^*)| \geqslant k_0$, and $|f(x^*) - Q_{j'}(x^*)| \to k_0$. But this implies that there exists a subsequence $\{j''\}$ of $\{j'\}$ such that $Q_{j''}(x^*) \to f(x^*) \pm k_0$, where $\pm$ denotes an appropriate sign $+$ or $-$ (if both are possible, choose one arbitrarily).

Denote now by $x_1, x_2, \ldots, x_n$ $(x_1 < x_2 < \cdots < x_n)$ the points of $J(a, b)$. Since $0 \leqslant |f(x) - Q_{j'}(x)| \leqslant |f(x^*) - Q_{j'}(x^*)| < \gamma_{j'}$ for each $x \in J(a, b)$, and $\gamma_{j'} \to k_0$, the sequence $\{j''\}$ has a subsequence $\{j_1'\}$ such that $\{|f(x_1) - Q_{j_1'}(x_1)|\}$ converges to some p_1, $0 \leqslant p_1 \leqslant k_0$. Hence, there exists a subsequence $\{j_1''\}$ of $\{j_1'\}$ such that $Q_{j_1''}(x_1) \to f(x_1) \pm p_1$. Using the sequence $\{j_1''\}$, the argument of this paragraph with x_1 replaced by x_2 defines a sequence $\{j_2''\}$ such that $Q_{j_2''}(x_i) \to f(x_i) \pm p_i$ for $i = 1, 2$. Similarly for $i = 3, 4, \ldots, n$; finally we obtain $\{j_n''\}$ such that $Q_{j_n''}(x_i) \to f(x_i) \pm p_i$ for $i = 1, 2, \ldots, n$, where $0 \leqslant p_i \leqslant k_0$. Observe that for some i, $p_i = k_0$.

For simplicity, let $\{j\}$ denote the sequence $\{j_n''\}$. Let $T(x)$ denote the Lagrange interpolation polynomial of degree $< n$ satisfying $T(x_i) = f(x_i) \pm p_i$, $i = 1, 2, \ldots, n$. Then $Q_j(x_i) \to T(x_i)$ for each $x_i \in J(a, b)$; and [2, Theorem 5] for each j and each $\gamma > 0$ there exists $Q_{j\gamma} \in Q(Z)$ such that

$$\| T(x) - Q_{j\gamma}(x)\|_\infty \leqslant \max_{J(a,b)} | T(x) - Q_j(x)| + \gamma.$$

Therefore, as $\gamma \to 0$, $\| T(x) - Q_{j\gamma}(x)\|_\infty \to 0$. Hence, $T(x)$ is approximable on $[a, b]$, and, therefore, its coefficients are integral [2, remark following Theorem 3]. Observe that $\max_{J(a,b)} | f(x) - T(x)| = k_0$.

We establish now the existence of the sequence $\{Q_l\}$ of the theorem. Let m be a positive integer satisfying $2/m < \max_{2 \leqslant i \leqslant n} (x_i - x_{i-1})$. Then, for each integer $k \geqslant m$, define a new function $f_k(x)$ as follows: (1) $f_k(x_i) = f(x_i) \pm p_i$ for each x_i; (2) $f_k(x) = f(x)$ for $x \in E_a \cup E \cup E_b$ where $E_a = [a, x_1 - (1/k)]$, $E_b = [x_n + (1/k), b]$, and $E = \bigcup_{i=2}^{n} [x_{i-1} + (1/k), x_i - (1/k)]$; and (3) $f_k(x)$ is linear on each of the $2n$ open intervals of $[a, b] - E_a - E_b - E - J(a, b)$, so that $f_k \in C(a, b)$. Then f_k converges to F at each $x \in [a, b]$. Also, for each k, f_k is approximable on $[a, b]$ (as the coefficients of T are integral), so that for every $\gamma > 0$ there exists $Q_{k,\gamma} \in Q(Z)$ such that $| f_k(x) - Q_{k,\gamma}(x)| < \gamma$ throughout $[a, b]$. For every $x \in [a, b]$, and, for $k = 1, 2,\ldots$, we have $Q_{k,1/k}(x) - F(x) = [Q_{k,1/k}(x) - f_k(x)] + [f_k(x) - F(x)] \to 0$ as $k \to \infty$. Hence, $\lim_{k\to\infty} Q_{k,1/k}(x) = F(x)$ for every $x \in [a, b]$. (We have assumed $n \geqslant 2$ and $a < x_1 < x_n < b$, but similar constructions can be made in the other cases).

DEFINITION 2. If f is approximable, let $F(x) \equiv f(x)$. If f is not approximable, let F be as in Theorem 1. For each $f \in C(a, b)$, we call F an apparent best approximation to f.

Note that, in general, F is not the restriction to $[a, b]$ of a $Q \in Q(Z)$, and, in fact, $F \in C(a, b)$ only if f is approximable. In addition, there may be more than one apparent best approximation to f if f is not approximable. For example, $f(x) \equiv \frac{1}{2}$ on $[-\frac{1}{2}, \frac{1}{2}]$ has $f_1(x) = \frac{1}{2}$ for $x \neq 0$, $f_1(0) = 1$, and $f_2(x) = \frac{1}{2}$ for $x \neq 0, f_2(0) = 0$ as apparent best approximations.

The significance of Theorem 1 is that, regardless of the questions of the existence of best approximations to f and the approximability of f, a convergence property holds in the following sense.

COROLLARY 1. *There always exists in $Q(Z)$ a sequence $\{Q_n\}$ converging to f in $[a, b] - J(a, b)$.*

With the previous theorem, the proofs of convergence theorems for other norms are greatly simplified.

DEFINITION 3. For $p \geqslant 1$ and $g \in C(a, b)$, we set

$$\| g \|_p = \left[\int_a^b | g(x)|^p \, dx\right]^{1/p}.$$

THEOREM 2. *There exists in $Q(Z)$ a sequence $\{Q_n\}$ such that $Q_n(x) \to f(x)$ in the sense that $\|f - Q_n \|_p \to 0$ as $n \to \infty$, for every p, $1 \leqslant p < \infty$.*

Proof. (i) If f is approximable, then for $n = 1, 2,\ldots$, there exists

$Q_n \in Q(Z)$ such that $|f(x) - Q_n(x)| < 1/n$ for all $x \in [a, b]$. Thus, $\|f - Q_n\|_p \leqslant 4/n$ for $1 \leqslant p < \infty$.

(ii) If f is not approximable, then by the proof of Theorem 1, for every $p \geqslant 1$, $|Q_{k,1/k}(x) - F(x)|^p \to 0$ and, hence, $\|F - Q_{k,1/k}\|_p \to 0$. Since $F = f$ almost everywhere in $[a, b]$, we have $\|f - Q_{k,1/k}\|_p \to 0$.

Theorem 2 tells us that there exist in $Q(Z)$ arbitrarily good approximations to f with respect to the L_p norms. Furthermore, the existence of a best approximation Q to f relative to such a norm would imply that $\|f - Q\|_p = 0$, and since $f, Q \in C(a, b)$, this would imply $f = Q$ on $[a, b]$.

COROLLARY 2. *There exists in $Q(Z)$ a best approximation to f relative to an L_p norm if and only if $f \in Q(Z)$.*

We study now the implications of Theorem 1 for a "discrete" norm.

DEFINITION 3. Let $p \geqslant 1$, and let $W = \{w_0, w_1, \ldots, w_l\}$ be a fixed set of distinct points in $[a, b]$. We set, for $Q \in Q(Z)$,

$$D_p(f, Q) = \left(\sum_{i=0}^{l} |f(w_i) - Q(w_i)|^p\right)^{1/p}, \qquad \text{and} \qquad C_w = J(a, b) \cap W.$$

By Theorem 1, every point w_i of W for which $|f(w_i) - Q(w_i)|$ cannot be made arbitrarily small by a judicious choice of Q must be in C_w.

THEOREM 3. *If either* (a) $C_w = \varnothing$, *or* (b) $C_w \neq \varnothing$ *and f is matchable on C_w, then there exist in $Q(Z)$ arbitrarily good approximations to f with respect to D_p.*

Proof. (a) Suppose $C_w = \varnothing$. Then for each i, $w_i \in [a, b] - J(a, b)$. By Corollary 1, there exists in $Q(Z)$ a sequence $\{Q_n\}$ such that, for every i, $Q_n(w_i) \to f(w_i)$. Therefore,

$$D_p(f, Q_n) = \left(\sum_{i=0}^{l} |f(w_i) - Q_n(w_i)|^p\right)^{1/p} \to 0.$$

(b) If $C_w \neq \varnothing$, and f is matchable on C_w, then there exists $\bar{Q} \in Q(Z)$ such that $\bar{Q}(x) = f(x)$ for all $x \in C_w$. Let m be as in the proof of Theorem 1; we again set $J(a, b) = \{x_1, x_2, \ldots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$, and again assume $n \geqslant 2$, $a < x_1 < x_n < b$. Similar arguments hold in the other cases. For each integer $k \geqslant m$, let $f_k(x)$ be

$\bar{Q}(x)$ for $x \in J(a, b)$,

$f(x)$ for $x \in [x_{i-1} + (1/k), x_i - (1/k)]$, $i = 2, 3, \ldots, n$,

$f(x)$ for $x \in [a, x_1 - (1/k)] \cup [x_n + (1/k), b]$,

linear on each of the remaining open subintervals of $[a, b]$.

Thus, each f_k is continuous in $[a, b]$, and, since it is matchable on $J(a, b)$, it is approximable on $[a, b]$ [12, Theorems 2.6 and 4.3]. For every $k \geqslant m$, $j \geqslant 1$, let $Q_{j,k} \in Q(Z)$ satisfy $\max_{a \leqslant x \leqslant b} |f_k(x) - Q_{j,k}(x)| < 1/j$. Since

$$f_k(x) \to \begin{cases} \bar{Q}(x) & \text{for } x \in J(a, b), \\ f(x) & \text{for } x \in [a, b] - J(a, b), \end{cases}$$

we have

$$Q_{k,k}(x) \to \begin{cases} \bar{Q}(x) & \text{for } x \in J(a, b), \\ f(x) & \text{for } x \in [a, b] - J(a, b). \end{cases}$$

In particular, it follows that $Q_{k,k}(w_i) \to f(w_i)$ for each i. Hence,

$$D_p(f, Q_{k,k}) = \left(\sum_{i=0}^{l} |f(w_i) - Q_{k,k}(w_i)|^p\right)^{1/p} \to 0, \qquad \text{Q.E.D.}$$

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